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Brill–Noether theory on Hirzebruch surfaces

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ABSTRACT

In this paper, we investigate higher rank Brill–Noether problems for stable vector bundles on Hirzebruch surfaces. Using suitable non-splitting extensions, we deal with the non-emptiness. Results concerning the emptiness follow as a consequence of a generalization of Clifford's theorem for line bundles on curves to vector bundles on surfaces.

1. Introduction

Let X be a smooth projective variety of dimension n over an algebraically closed field K of characteristic 0 and let $M_H = M_{X,H}(r; c_1, \dots, c_s)$ be the moduli space of rank r vector bundles E on X stable with respect to an ample line bundle H and with fixed Chern classes $c_i(E) = c_i$ for $i = 1, \dots, s := min\{r, n\}$. A Brill–Noether locus $W^k_H(r; c_1, \dots, c_s)$ of M_H is a subvariety of $M_{X,H}(r; c_1, \dots, c_s)$ whose points correspond to stable vector bundles having at least k independent sections and, roughly speaking, the Brill–Noether theory describes the geometry of these varieties. The kernel of the idea of the classical Brill–Noether theory is found at least as early as the work of Brill and Noether in the 19th century and deals with line bundles on algebraic curves (see [1]). Its first natural generalization concerns higher rank vector bundles on algebraic curves and in this context, the theory has been extensively developed during the last decades by several authors (see the overview [10] and references quoted there). Very recently, the foundation of a generalized Brill–Noether theory for moduli space of rank $r \geq 2$ stable vector bundles on higher-dimensional varieties has been formulated [3,17,18,20] and very interesting problems have been settled. However, basic questions as whether the Brill–Noether locus $W^k_H(r; c_1, \dots, c_s)$ is non-empty, irreducible or if it has the expected dimension remain unclear and the great amount of properties and pathologies that appear in this more general context (see for example [3]) makes this new theory an emerging field of interest.

In [3] and as a natural generalization of the classical Brill–Noether theory, we define the Brill–Noether locus $W^k_H(r; c_1, \dots, c_s)$ in $M_{X,H}(r; c_1, \dots, c_s)$ as the set of stable vector bundles in $M_{X,H}(r; c_1, \dots, c_s)$ having at least k independent sections and we prove that $W^k_H(r; c_1, \dots, c_s)$ has a natural structure of scheme provided certain cohomological groups vanish. These cohomological assumptions are natural if we want to have a filtration of the moduli space M_H by the subvarieties $W^k_H(r; c_1, \dots, c_s)$. Indeed, if X is an n-dimensional projective variety, then any vector bundle E on X has $n + 1$ cohomological groups whose dimensions are related by the Riemann–Roch theorem and one is forced to look for a multigraded filtration of the moduli space M_H by means of the sets $\{E \in M_H | h^i(E) \geq k_i\}$. Under the cohomological assumptions, $h^i(E) = 0$ for $i \geq 2$ and for any $E \in M_H$, the only non-vanishing cohomology groups are $H^0(E)$ and $H^1(E)$ and their dimensions are subject to one relation given by the Riemann–Roch theorem: $\dim H^0(E) - \dim H^1(E) = \chi(E) = \chi(r; c_1, \dots, c_s)$. Hence, it makes sense to consider only the filtration of the moduli space M_H by the dimension of the space of global sections. In this paper, we consider the Brill–Noether theory for vector bundles on Hirzebruch surfaces where the above cohomological restrictions are automatically satisfied (see Section 2). More precisely, we will mainly be concerned with the questions whether the Brill–Noether loci are empty or non-empty. For the non-emptiness, our key strategy is to

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deal with suitable prioritary vector bundles. Prioritary vector bundles were introduced by Hirschowitz and Laszlo [15] in the context of vector bundles on $\mathbb{P}^2$ and later on by Walter [21] in the context of rational surfaces. We will use them to construct via extensions stable vector bundles with a fix number of independent sections and, hence, to prove the existence of non-empty irreducible components of the Brill–Noether locus with the expected dimension. In the classical Brill–Noether theory for line bundles on curves (as well for higher rank bundles on curves), Clifford's theorem, which bounds the number of independent global sections, plays an important role. Indeed, Clifford's theorem together with the Riemann–Roch theorem is basic to bound the so-called Brill–Noether region. In this paper, we have generalized Clifford's theorem for vector bundles on curves to vector bundles on surfaces and this result will be the key point for proving our main result concerning the emptiness of the Brill–Noether locus.

Next we outline the structure of this paper. In Section 2, we recall the basic facts on Hirzebruch surfaces, prioritary sheaves and stable sheaves, as well as, the scheme-theoretic construction of the Brill–Noether loci defined in the moduli space of stable vector bundles on Hirzebruch surfaces. Section 3 is devoted to study the non-emptiness of the Brill–Noether loci of stable vector bundles on Hirzebruch surfaces and it contains the main results of this paper. In this section, we start a systematic study of the Brill–Noether loci $W^k_H(r; c_1, c_2)$ of the moduli space $M_H(r; c_1, c_2)$ of stable vector bundles on a Hirzebruch surface and our goal is to determine sufficient conditions for the non-emptiness of $W^k_H(r; c_1, c_2)$ (see Theorem 3.2).

Section 4 is concerned with the emptiness of the Brill–Noether loci of stable vector bundles on Hirzebruch surfaces. We first generalize Clifford's theorem for line bundles on curves to vector bundles on surfaces and as a nice application we obtain our results about emptiness (see Corollaries 4.2 and 4.3).

Notation. We will work over an algebraically closed field K of characteristic zero. Let X be a smooth projective surface and let E be a rank r vector bundle on X with Chern classes $c_i(E) = c_i$, $i = 1, 2$. We will write $h^i(E)$ (resp. $ext^i(E, F)$) to denote the dimension of the ith cohomology group $H^i(X, E) = H^i(E)$ (resp. ith Ext group $\mathrm{Ext}^i(E, F)$) as a K-vector space. Set $\chi(r; c_1, c_2) := \chi(E) := h^0(E) - h^1(E) + h^2(E)$. We will denote by K_X the canonical divisor on X.

2. Preliminaries

The goal of this section is to collect the results concerning Hirzebruch surfaces, prioritary sheaves and stable sheaves that we will use through this paper and to recall the construction and main properties of the Brill–Noether loci defined in the moduli space of stable vector bundles on Hirzebruch surfaces.

- Hirzebruch surfaces:

For any integer $e \geq 0$, let $X_e \cong \mathbb{P}(\mathcal{E}) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$ be a non-singular Hirzebruch surface. We denote by C_0 and F the standard basis of $\mathrm{Pic}(X_e) \cong \mathbb{Z}^2$ such that $C_0^2 = -e$, $F^2 = 0$ and $C_0F = 1$. The canonical divisor is given by

$$K_{X_e} = -2C_0 - (e+2)F$$

and $\chi(\mathcal{O}_{X_e}) = 1$.

Remark 2.1. It is well known that a divisor $L = aC_0 + bF$ on X_e is ample if and only if it is very ample, if and only if $a > 0$ and $b > ae$; and that $D = a'C_0 + b'F$ is effective if and only if $a' \geq 0$ and $b' \geq 0$ ([11]; V, Corollary 2.18).

Moreover, the following holds:

Lemma 2.2. *Let $X_e = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$ be a Hirzebruch surface. For any line bundle $\mathcal{O}_{X_e}(aC_0 + bF)$ on X_e we have*

$$H^i(X_e, \mathcal{O}_{X_e}(aC_0 + bF)) = \begin{cases} 0 & \text{if } a = -1 \\ H^i(\mathbb{P}^1, S^a(\mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}^1}(b)) & \text{if } a \geq 0 \\ H^{2-i}(\mathbb{P}^1, S^{-2-a}(\mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}^1}(-e-b-2))^* & \text{if } a \leq -2 \end{cases}$$

where $S^a(\mathcal{E})$ denotes the a-th symmetric power of $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)$.

Proof. See [7]; Lemma 2.9. □

- Prioritary and stable sheaves:

We consider X a smooth projective surface, H an ample divisor on X, $r \geq 2$ an integer and $c_i \in H^{2i}(X, \mathbb{Z})$ for $i = 1, 2$. We denote by $M_H(r; c_1, c_2)$ the coarse moduli space of rank r vector bundles E on X, with Chern classes $c_i(E) = c_i$ for $i = 1, 2$, and H-stable according to the following definition due to Mumford and Takemoto.

Definition 2.3. Let H be an ample line bundle on a smooth projective surface X. For a torsion free sheaf G on X we set

$$\mu(G) = \mu_H(G) := \frac{c_1(G)H}{rk(G)}.$$

The sheaf G is said to be H-*semistable* if

$$\mu_H(E) \leq \mu_H(G)$$

for all non-zero subsheaves $E \subset G$ with $rk(E) < rk(G)$; if strict inequality holds then G is H-*stable*. Notice that for rank r vector bundles G on X with $(c_1(G)H, r) = 1$, the concepts of H-stability and H-semistability coincide.

In the next result, we have summarized well-known properties of moduli spaces of stable rank r vector bundles on projective surfaces that we will need later on (for a general reference see [15] and the papers quoted there).

Proposition 2.4. *Let X be a smooth, projective surface, H an ample divisor on X, $c_1 \in H^2(X,\mathbb{Z})$ and $c_2 \in \mathbb{Z}$. Then for $c_2 \gg 0$, the moduli space $M_{X,H}(r; c_1, c_2)$ of rank r, H-stable vector bundles E on X with fixed Chern classes $c_i(E) = c_i$ is a non-empty generically smooth, irreducible, quasiprojective variety of the expected dimension*

$$\dim(M_{X,H}(r; c_1, c_2)) = 2rc_2 - (r-1)c_1^2 - (r^2-1)\chi(\mathcal{O}_X).$$

Remark 2.5. For birationally ruled surfaces and, in particular, for Hirzebruch surfaces, the moduli space $M_{X,H}(r; c_1, c_2)$ is not only generically smooth but smooth everywhere ([21]; Theorem 1).

Although we want to study the geometry of the moduli space of stable vector bundles on a Hirzebruch surface in terms of their subvarieties, we will need in our approach to consider not only stable vector bundles but prioritary sheaves. Prioritary sheaves were introduced on $\mathbb{P}^2$ (resp. on birationally ruled surfaces) by Hirschowitz–Laszlo (resp. Walter) as a generalization of semistable sheaves. (The reader can see [15,21] for more information on prioritary sheaves). Let us start recalling the definition of prioritary sheaf on a Hirzebruch surface.

Definition 2.6. Let X_e be a Hirzebruch surface. A coherent sheaf E on X_e is said to be *prioritary* if it is torsion free and if $\mathrm{Ext}^2(E, E(-F)) = 0$.

The following Lemma is the key point in order to be able to use prioritary sheaves to study moduli spaces of stable vector bundles.

Lemma 2.7. *Let X_e be a Hirzebruch surface and let H be an ample divisor on X_e. Then any H-semistable, torsion free sheaf E on X_e is prioritary.*

Proof. If E is an H-semistable torsion free sheaf on X_e, then any non-zero torsion free quotient Q of E would have H-slope satisfying $\mu_H(Q) \geq \mu_H(E)$, while any non-zero subsheaf S of E would have H-slope satisfying $\mu_H(S) \leq \mu_H(E)$. If E were not prioritary, there would exist a non-zero homomorphism

$$\phi \in \mathrm{Hom}(E, E(K_{X_e} + F)) \cong \mathrm{Ext}^2(E, E(-F))^*.$$

Since a twist of a H-semistable sheaf is H-semistable, the image of ϕ would then satisfy

$$\mu_H(E) \leq \mu_H(im(\phi)) \leq \mu_H(E(K_{X_e} + F)) = \mu_H(E) + H(K_{X_e} + F),$$

contradicting the fact that for any ample divisor H on X_e we have $H(K_{X_e} + F) < 0$. □

Remark 2.8. We denote by $Sp(r; c_1, c_2)$ the coarse moduli space of rank r simple prioritary torsion free sheaves E on X_e, with Chern classes $c_i(E) = c_i$ for $i = 1, 2$. According to [9]; pages 195–196 and [21], the moduli space $Sp(r; c_1, c_2)$ is smooth and irreducible of the expected dimension $2rc_2 - (r-1)c_1^2 - (r^2-1)$.

- Brill–Noether loci:

Let X_e be a Hirzebruch surface, H an ample line bundle on X_e, $c_1 \in H^2(X_e,\mathbb{Z})$ and $c_2 \in \mathbb{Z}$. Contained in the moduli space $M_H(r; c_1, c_2)$ there is the Brill–Noether locus $W_H^k(r; c_1, c_2)$ defined set theoretically as the subset of $M_H(r; c_1, c_2)$ whose points correspond to vector bundles having at least k independent sections. The Brill–Noether theory describes the geometry of $W_H^k(r; c_1, c_2)$ and basic questions such as when $W_H^k(r; c_1, c_2)$ is non-empty, what is its dimension and whether it is irreducible are subtle and of great interest. We will start with a scheme-theoretic description of $W_H^k(r; c_1, c_2)$. We will prove that $W_H^k(r; c_1, c_2)$ is a (locally) determinantal variety which justifies its expected dimension and singular locus.

Notice that if an H-stable vector bundle E on X_e has some non-zero section, then $c_1(E)H \geq 0$. So, without loss of generality we can assume that $c_1(E)H \geq 0$ since otherwise $W_H^k(r; c_1, c_2)$ is empty.

First of all we assume that $M_H = M_H(r; c_1, c_2)$ is a fine moduli space. Let $\mathcal{U} \to X_e \times M_H$ be a universal family such that for any $t \in M_H$, $\mathcal{U}|_{X_e \times \{t\}} = E_t$ is an H-stable, rank r vector bundle on X_e with Chern classes $c_i(E_t) = c_i$. Let D be an effective divisor on X_e such that for any $t \in M_H$,

$$h^0(X_e, E_t(D)) = \chi(E_t(D)) \quad \text{and} \quad H^i(X_e, E_t(D)) = 0, \quad i \geq 1. \tag{2.1}$$

We consider $\mathcal{D} = D \times M_H$ the corresponding product divisor on $X_e \times M_H$ and we denote by

$$\nu : X_e \times M_H \to M_H$$

the natural projection. It follows from (2.1) and the base change theorem that $\nu_*\mathcal{U}(\mathcal{D})$ is a locally free sheaf of rank $\chi(E_t(D))$ on M_H and

$$R^i\nu_*\mathcal{U}(\mathcal{D}) = 0, \quad i > 0.$$

Therefore, applying the functor ν_* to the short exact sequence

$$0 \to \mathcal{U} \to \mathcal{U}(\mathcal{D}) \to \mathcal{U}(\mathcal{D})/\mathcal{U} \to 0$$

we get the following exact sequence

$$0 \to \nu_* \mathcal{U} \to \nu_* \mathcal{U}(\mathcal{D}) \xrightarrow{\gamma} \nu_*(\mathcal{U}(\mathcal{D})/\mathcal{U}) \to R^1 \nu_* \mathcal{U} \to 0.$$

The map γ is a morphism between locally free sheaves on M_H of rank $\chi(E_t(D))$ and $\chi(E_t(D)) - \chi(E)$ respectively and the $(\chi(E_t(D)) - k)$-th determinantal variety

$$W_H^k(r; c_1, c_2) \subset M_H$$

associated to it has support

$$\{E_t \in M_H | \mathrm{rank}\gamma_{E_t} \leq \chi(E_t(D)) - k\}$$

i.e. $W_H^k(r; c_1, c_2)$ is the locus where the fibre of $R^1 \nu_* \mathcal{U}$ has dimension at least $(\chi(E_t(D)) - \chi(E_t)) - (\chi(E_t(D)) - k) = k - \chi(E_t)$. For any $E_t \in M_H$, $h^2(X_e, E_t) = 0$. Indeed, if $h^2(X_e, E_t) > 0$, by Serre duality $h^0(E_t^*(K_{X_e})) > 0$ and since E_t is H-stable the same is true for E_t^* and thus this inequality implies that

$$-K_{X_e} H < \frac{c_1(E_t^*)H}{r} = \frac{-c_1 H}{r} \leq 0$$

which is a contradiction since $-K_{X_e}$ is an effective divisor.

Therefore,

$$h^1(E_t) = h^0(E_t) - \chi(E_t).$$

Thus,

$$\begin{aligned} Supp(W_H^k(r; c_1, c_2)) &= \{E \in M_H | h^1(E) \geq k - \chi(E)\} \\ &= \{E \in M_H | h^0(E) \geq k\}. \end{aligned}$$

Moreover, since $W_H^k(r; c_1, c_2)$ is a $(\chi(E_t(D)) - k)$-determinantal variety associated to a morphism between locally free sheaves of rank $\chi(E_t(D))$ and $\chi(E_t(D)) - \chi(E)$ respectively, any of its non-empty irreducible components has dimension greater or equal to

$$\dim(M_H) - k(k - \chi(E))$$

and

$$W_H^{k+1}(r; c_1, c_2) \subset Sing(W_H^k(r; c_1, c_2))$$

whenever $W_H^k(r; c_1, c_2) \neq M_{X,H}(r; c_1, c_2)$.

If $M_H(r; c_1, c_2)$ is not a fine moduli space, it is also possible to carry out the construction of the Brill–Noether locus using only the local existence of a universal sheaf on $M_H(r; c_1, c_2)$. Indeed, as in [10] we can carry out the constructions locally, show the independence of the choice of the locally universal sheaf and conclude that the construction glue as a global algebraic object.

Summing up, we have the following result

Theorem 2.9. *Let X_e be a Hirzebruch surface and let $M_H(r; c_1, c_2)$ be a moduli space of rank r H-stable vector bundles E on X_e with fixed Chern classes $c_i(E) = c_i$. Then, for any $k \geq 0$, there exists a determinantal variety $W_H^k(r; c_1, c_2)$ such that*

$$Supp(W_H^k(r; c_1, c_2)) = \{E \in M_H(r; c_1, c_2) | h^0(E) \geq k\}.$$

Moreover, each non-empty irreducible component of $W_H^k(r; c_1, c_2)$ has dimension greater than or equal to

$$\begin{aligned} \rho_H^k(r; c_1, c_2) &= \dim M_H(r; c_1, c_2) - k(k - \chi(r; c_1, c_2)) \\ &= \dim M_H(r; c_1, c_2) - k\left(k - r + \frac{c_1 K_{X_e}}{2} - \frac{c_1^2}{2} + c_2\right) \end{aligned}$$

and $W_H^{k+1}(r; c_1, c_2) \subset Sing(W_H^k(r; c_1, c_2))$ whenever $W_H^k(r; c_1, c_2) \neq M_H(r; c_1, c_2)$.

This result led us to pose in [3] the following definition

Definition 2.10. The variety $W_H^k(r; c_1, c_2)$ is called the *k-Brill–Noether locus* of the moduli space $M_H(r; c_1, c_2)$ (or simply the Brill–Noether locus if there is no confusion) and

$$\rho_H^k(r; c_1, c_2) := \dim M_H(r; c_1, c_2) - k(k - \chi(r; c_1, c_2))$$

is called the *generalized Brill–Noether number* where by definition $\chi(r; c_1, c_2) = \chi(E)$, for any $E \in M_H(r; c_1, c_2)$. By the Riemann–Roch theorem, $\chi(r; c_1, c_2)$ can be expressed as

$$\chi(r; c_1, c_2) := r - \frac{c_1 K_{X_e}}{2} + \frac{c_1^2}{2} - c_2.$$

Remark 2.11. Under the assumption $c_2 \gg 0$, the moduli space $M_H(r; c_1, c_2)$ has the expected dimension given in Proposition 2.4 and hence

$$\rho_H^k(r; c_1, c_2) = 2rc_2 - (r-1)c_1^2 - (r^2-1) - k(k - \chi(r; c_1, c_2)).$$

Notation 2.12. If there is no confusion then, we will simply write W^k and ρ^k instead of $W_H^k(r; c_1, c_2)$ and $\rho_H^k(r; c_1, c_2)$.

By Theorem 2.9, the Brill–Noether locus $W_H^k(r; c_1, c_2)$ has dimension greater or equal to $\rho_H^k(r; c_1, c_2)$ and the number $\rho_H^k(r; c_1, c_2)$ is also called the expected dimension of the corresponding Brill–Noether locus. Hence, we are led to pose the question whether the dimension of the Brill–Noether locus $W_H^k(r; c_1, c_2)$ and its expected dimension coincide provided the Brill–Noether locus $W_H^k(r; c_1, c_2)$ is non-empty. More precisely, we have

Question 2.13. Let X_e be a Hirzebruch surface and H an ample divisor on X_e

(1) Does $\rho_H^k(r; c_1, c_2) < 0$ imply $W_H^k(r; c_1, c_2) = \emptyset$?
(2) Does $\rho_H^k(r; c_1, c_2) \geq 0$ imply $W_H^k(r; c_1, c_2) \neq \emptyset$?
(3) Do $\rho_H^k(r; c_1, c_2) \geq 0$ and $W_H^k(r; c_1, c_2) \neq \emptyset$ imply

$$\rho_H^k(r; c_1, c_2) = \dim W_H^k(r; c_1, c_2)?$$

In subsequent sections, we will address the problem of the non-emptiness (resp. emptiness) of the Brill–Noether loci for the moduli space of stable vector bundles on Hirzebruch surfaces. The reader can look at [3] for examples of non-empty Brill–Noether loci with the negative generalized Brill–Noether number and examples where the dimension of the Brill–Noether locus and the expected dimension of the Brill–Noether locus do not coincide in spite of being both positive (See [3]; Example 2.11).

3. Non-emptiness of the Brill–Noether loci of stable vector bundles on Hirzebruch surfaces

In [4–7], the authors studied the moduli space of stable vector bundles on Hirzebruch surfaces but very little is known about their geometry in terms of the existence and structure of their subvarieties. In this section we will focus our attention on the Brill–Noether stratification of moduli spaces of stable vector bundles on Hirzebruch surfaces and we will investigate its non-emptiness. More precisely, we fix integers $0 < x \in \mathbb{Z}$ and $0 \ll c_2 \in \mathbb{Z}$ and an ample divisor $H = C_0 + (x+e+1)F$ and we study the Brill–Noether loci $W_H^k(r; C_0 - xF, c_2)$ of the moduli space $M_H(r; C_0 - xF, c_2)$ of H-stable rank r vector bundles E on a Hirzebruch surface X_e with Chern classes $c_1(E) = C_0 - xF$ and $c_2(E) = c_2$.

Remark 3.1. Notice that under the above assumptions $x > 0$ and $c_2 \gg 0$, we have $\rho_H^k(r; C_0 - xF, c_2) < 0$ for $k \geq 2r$.

According to this remark, it is natural to study the Brill–Noether loci $W_H^k(r; C_0 - xF, c_2)$ of the Brill–Noether stratification of the moduli space $M_H(r; C_0 - xF, c_2)$ for k in the range $1 \leq k \leq 2r - 1$. We have got the following

Theorem 3.2. *Let X_e be a Hirzebruch surface, x, r, c_2 integers with $x > 0$, $r \geq 2$ and $c_2 \gg 0$ and let $H := C_0 + (x+e+1)F$ be an ample divisor on X_e. Then for any integer k, $1 \leq k \leq r-2$ such that $c_2 \equiv 0 \mod (r-k-1)$ there exists an irreducible component of the Brill–Noether loci $W_H^k(r; C_0 - xF, c_2)$ of the moduli space $M_H(r; C_0 - xF, c_2)$ which has the expected dimension, namely,*

$$\rho_H^k(r; C_0 - xF, c_2) = 2rc_2 - (r-k-1)(-e-2x) - kc_2 - r^2 + 1 - k^2 + k(r+1).$$

To this end, we need the following key result

Proposition 3.3. *Let X_e be a Hirzebruch surface and let H be an ample divisor on X_e. Fix integers α, x and c with $\alpha, x > 0$ and $c < 0$. If $c \ll 0$, then the moduli space $M_H(\alpha+1; C_0 - xF, -\alpha c)$ is a non-empty smooth irreducible variety of dimension*

$$-2(\alpha+1)\alpha c + \alpha(e+2x) - (\alpha+1)^2 + 1$$

and a generic vector bundle $E \in M_H(\alpha+1; C_0 - xF, -\alpha c)$ sits in an exact sequence of the following type

$$0 \to \mathcal{O}_{X_e}(C_0 - (x - \alpha c)F) \to E \to \mathcal{O}_{X_e}(-cF)^{\alpha} \to 0. \tag{3.1}$$

Proof. It follows from Proposition 2.4 and Remark 2.5 that for $c \ll 0$, the moduli space $M_H(\alpha+1; C_0 - xF, -\alpha c)$ is a non-empty smooth irreducible variety of the stated dimension. The non-trivial part is to prove that a generic vector bundle $E \in M_H(\alpha+1; C_0 - xF, -\alpha c)$ sits in an exact sequence of the type (3.1). To this end, we consider $V = \mathrm{Ext}^1(\mathcal{O}_{X_e}(-cF), \mathcal{O}_{X_e}(C_0 - (x-\alpha c)F))$ and $Gr_\alpha = Gr(\alpha, V)$ be the Grassmannian which parametrizes α-dimensional linear subspaces of V. Any element $y \in Gr_\alpha$ corresponds to an extension:

$$0 \to \mathcal{O}_{X_e}(C_0 - (x - \alpha c)F) \to E_y \to \mathcal{O}_{X_e}(-cF)^{\alpha} \to 0. \tag{3.2}$$

Moreover, there exists a universal extension

$$0 \to q^*\mathcal{O}_{X_e}(C_0 - (x - \alpha c)F) \otimes p^*\mathcal{O}_{Gr_\alpha}(1) \to \mathcal{E} \to q^*\mathcal{O}_{X_e}(-cF)^{\alpha} \to 0 \tag{3.3}$$

on the product $Gr_\alpha \times X_e$ (with projections p and q to Gr_α and X_e, respectively), such that for each rational point $[y] \in Gr_\alpha$, the fibre $\mathcal{E}_y = \mathcal{E}_{|\{y\} \times X_e}$ is isomorphic to E_y (see [15]; Example 2.1.12).

Claim. *We have:*

(a) *For any rational point* $[y] \in Gr_\alpha$, E_y *is a rank* $(\alpha + 1)$ *simple, prioritary vector bundle on* X_e *with Chern classes*

$$(c_1(E_y), c_2(E_y)) = (C_0 - xF, -\alpha c).$$

(b) Gr_α *is a non-empty irreducible rational variety, and*

$$\dim Gr_\alpha = -2(\alpha + 1)\alpha c + \alpha(e + 2x) - (\alpha + 1)^2 + 1.$$

(c) *There is an open embedding*

$$\phi : Gr_\alpha \longrightarrow Sp(\alpha + 1; C_0 - xF, -\alpha c)$$

where $Sp(\alpha + 1; C_0 - xF, -\alpha c)$ *denotes the moduli space of simple prioritary rank* $(\alpha + 1)$ *torsion free sheaves* E *on* X_e *with fixed Chern classes* $c_1(E) = C_0 - xF$ *and* $c_2(E) = -\alpha c$.

Proof of the Claim. (a) It easily follows from the construction that E_y is a rank $(\alpha + 1)$ vector bundle on X_e with Chern classes

$$(c_1(E_y), c_2(E_y)) = (C_0 - xF, -\alpha c).$$

Let us show that E_y is a prioritary vector bundle. Since E_y is a torsion free sheaf, we only have to check that $\mathrm{Ext}^2(E_y, E_y(-F)) = 0$. Applying the functor $\mathrm{Hom}(E_y, .)$ to the exact sequence (3.2) twisted by $\mathcal{O}_{X_e}(-F)$, we get the long exact sequence

$$\cdots \to \mathrm{Ext}^2(E_y, \mathcal{O}_{X_e}(C_0 - (x - \alpha c + 1)F)) \to \mathrm{Ext}^2(E_y, E_y(-F)) \\ \to \mathrm{Ext}^2(E_y, \mathcal{O}_{X_e}((-c - 1)F)^\alpha) \to 0.$$

By Serre's duality and using again the exact sequence (3.2), we get

$$\mathrm{Ext}^2(E_y, \mathcal{O}_{X_e}(C_0 - (x - \alpha c + 1)F) = H^0 E_y(-C_0 + (x - \alpha c + 1)F + K_{X_e})^* = 0,$$
$$\mathrm{Ext}^2(E_y, \mathcal{O}_{X_e}((-c - 1)F)) = H^0 E_y(K_{X_e} + (c + 1)F)^* = 0,$$

where K_{X_e} is the canonical divisor of X_e. Thus, $\mathrm{Ext}^2(E_y, E_y(-F)) = 0$ and E_y is a prioritary vector bundle.

Next we will see that E_y is simple, i.e., $hom(E_y, E_y) = 1$. Applying the functor $\mathrm{Hom}(., E_y)$ to the exact sequence (3.2) we get the following long exact sequence

$$0 \to \mathrm{Hom}(\mathcal{O}_{X_e}(-cF)^\alpha, E_y) \to \mathrm{Hom}(E_y, E_y) \to \mathrm{Hom}(\mathcal{O}_{X_e}(C_0 - (x - \alpha c)F), E_y) \to \cdots.$$

Since $H^0\mathcal{O}_{X_e}(-C_0 + (x - \alpha c - c)F) = 0$, from the exact sequence (3.2) we deduce

$$hom(\mathcal{O}_{X_e}(C_0 - (x - \alpha c)F), E_y) = h^0 E_y(-C_0 + (x - \alpha c)F) = 1.$$

Consider the long exact cohomology sequence

$$0 \to H^0\mathcal{O}_{X_e}(C_0 - (x - \alpha c - c)F) \to H^0 E_y(cF) \to H^0\mathcal{O}_{X_e}^\alpha \xrightarrow{\delta} H^1\mathcal{O}_{X_e}(C_0 - (x - \alpha c - c)F) \to$$

associated to the exact sequence

$$0 \to \mathcal{O}_{X_e}(C_0 - (x - \alpha c - c)F) \to E_y(cF) \to \mathcal{O}_{X_e}^\alpha \to 0.$$

Since $H^1\mathcal{O}(C_0 - (x - \alpha c - c)F) = \mathrm{Ext}^1(\mathcal{O}_{X_e}(-cF), \mathcal{O}_{X_e}(C_0 - (x - \alpha c)F))$, the map

$$\delta : H^0\mathcal{O}_{X_e}^\alpha \longrightarrow H^1\mathcal{O}_{X_e}(C_0 - (x - \alpha c - c)F)$$

is an injection. This, together with the fact that, by Remark 2.1, $H^0\mathcal{O}_{X_e}(C_0 - (x - \alpha c - c)F) = 0$, gives us

$$H^0 E_y(cF) = 0.$$

Therefore, $hom(E_y, E_y) = 1$ which proves (a).

(b) Since Gr_α is the Grassmannian variety of α-dimensional linear subspaces of $V = \mathrm{Ext}^1(\mathcal{O}_{X_e}(-cF), \mathcal{O}_{X_e}(C_0-(x-\alpha c)F))$, it is a non-empty irreducible rational variety. Let us compute its dimension. Setting

$$s = ext^1(\mathcal{O}_{X_e}(-cF), \mathcal{O}_{X_e}(C_0 - (x - \alpha c)F)) = h^1\mathcal{O}_{X_e}(C_0 - (x - (\alpha + 1)c)F)$$

we have

$$\dim Gr_\alpha = \dim Gr(\alpha, V) = \alpha(s - \alpha).$$

Since $x > 0$ and $c < 0$, by Remark 2.1 and Serre duality we get

$$H^0\mathcal{O}_{X_e}(C_0 - (x - (\alpha + 1)c)F) = H^2\mathcal{O}_{X_e}(C_0 - (x - (\alpha + 1)c)F) = 0.$$

Therefore, applying the Riemann–Roch theorem we obtain

$$\begin{aligned} s &= -\chi(\mathcal{O}_{X_e}(C_0-(x-(\alpha+1)c)F)) \\ &= -1+\frac{(C_0-(x-(\alpha+1)c)F)K_{X_e}}{2}-\frac{(C_0-(x-(\alpha+1)c)F)^2}{2} \\ &= 2(x-(\alpha+1)c)+e-2. \end{aligned}$$

Therefore,

$$\begin{aligned} \dim Gr_\alpha = \alpha s-\alpha^2 &= 2\alpha x-2(\alpha+1)\alpha c+\alpha(e-2)-\alpha^2 \\ &= -2(\alpha+1)\alpha c+\alpha(e+2x)-(\alpha+1)^2+1. \end{aligned}$$

(c) Let us see that the induced map

$$\phi : Gr_\alpha \longrightarrow Sp(\alpha+1; C_0-xF, -\alpha c)$$

is an open embedding. Assume that there are two rational points $[y], [y'] \in Gr_\alpha$ such that the corresponding extensions

$$\begin{aligned} &0 \to \mathcal{O}_{X_e}(C_0-(x-\alpha c)F) \xrightarrow{\beta_1} E_y \xrightarrow{\beta_2} \mathcal{O}_{X_e}(-cF)^\alpha \to 0 \\ &0 \to \mathcal{O}_{X_e}(C_0-(x-\alpha c)F) \xrightarrow{\gamma_1} E_{y'} \xrightarrow{\gamma_2} \mathcal{O}_{X_e}(-cF)^\alpha \to 0 \end{aligned}$$

are isomorphic (i.e. $E_y \cong E_{y'}$). Since

$$hom(\mathcal{O}_{X_e}(C_0-(x-\alpha c)F), \mathcal{O}_{X_e}(-cF)) = h^0\mathcal{O}_{X_e}(-C_0+(x-\alpha c-c)F)=0,$$

we get $\beta_2\circ\gamma_1=\gamma_2\circ\beta_1=0$. Thus, there exists $\lambda \in Aut(\mathcal{O}_{X_e}(C_0-(x-\alpha c)F)) \cong K$ such that $\gamma_1=\beta_1\circ\lambda$ and hence ϕ is an injection. By Remark 2.8, we know that for any simple prioritary sheaf E the moduli space $Sp(\alpha+1; C_0-xF, -\alpha c)$ is smooth at $[E]$ of the expected dimension. So, $Sp(\alpha+1; C_0-xF, -\alpha c)$ is smooth along $\phi(Gr_\alpha)$ of dimension

$$-2(\alpha+1)\alpha c+\alpha(e+2x)-(\alpha+1)^2+1.$$

On the other hand, this is also $\dim Gr_\alpha$. As also Gr_α is smooth, the injective morphism ϕ is an open embedding and this completes the proof of the claim. $\square$

As we pointed out at the beginning of the proof for $c \ll 0$, the moduli space

$$M_H(\alpha+1; C_0-xF, -\alpha c)$$

is a non-empty smooth irreducible variety of dimension

$$-2(\alpha+1)\alpha c+\alpha(e+2x)-(\alpha+1)^2+1.$$

By Lemma 2.7, the moduli space $M_H(\alpha+1; C_0-xF, -\alpha c)$ is an open subspace of the moduli space $Sp(\alpha+1; C_0-xF, -\alpha c)$ which by Remark 2.8 is smooth and irreducible of the same dimension. This together with the fact that ϕ is an open embedding implies that a generic vector bundle $E \in M_H(\alpha+1; C_0-xF, -\alpha c)$ sits in an exact sequence of the desired type

$$0 \to \mathcal{O}_{X_e}(C_0-(x-\alpha c)F) \to E \to \mathcal{O}_{X_e}(-cF)^\alpha \to 0. \quad \square \tag{3.4}$$

Notation 3.4. From now on we will denote by $\mathcal{H}^0_H(\alpha+1; C_0-xF, -\alpha c)$ the open subset of $M_H(\alpha+1; C_0-xF, -\alpha c)$ parametrizing H-stable vector bundles given by an extension of type (3.1).

Now we are ready to prove the main result of this section

Proof of Theorem 3.2. Fix k an integer, $1 \le k \le r-2$, such that $(r-k-1)$ divides c_2 and denote by f the negative integer such that $c_2=-(r-k-1)f$. Let $\mathcal{F}$ be the set of rank r vector bundles E on X_e given by a non-trivial extension

$$0 \to \mathcal{O}^k_{X_e} \to E \to G \to 0 \tag{3.5}$$

where G sits in the open subset $\mathcal{H}^0_H := \mathcal{H}^0_H(r-k; C_0-xF, c_2) \subset M_H(r-k; C_0-xF, c_2)$ given by Proposition 3.3 and Notation 3.4 taking there $\alpha=r-k-1$ and $c=f$. Let us endow $\mathcal{F}$ with a natural structure of scheme. To this end, we consider $\mathcal{A}^0 \longrightarrow \mathcal{H}^0_H \times X_e$ the Poincaré sheaf of H-stable vector bundles on X_e such that for any $h \in \mathcal{H}^0_H$, $\mathcal{A}^0|_{\{h\}\times X_e}$ is isomorphic to the bundle E of $\mathcal{H}^0_H$ corresponding to $h \in \mathcal{H}^0_H$. Let $\pi : X_e \times \mathcal{H}^0_H \longrightarrow \mathcal{H}^0_H$ and $p : X_e \times \mathcal{H}^0_H \longrightarrow X_e$ the natural projections.

Set $\mathcal{E} := \mathrm{Ext}^1_\pi(\mathcal{A}^0, p^*\mathcal{O}_{X_e})$. $\mathcal{E}$ is a locally free sheaf on $\mathcal{H}^0_H$ of rank $n=\dim \mathrm{Ext}^1(G, \mathcal{O}_{X_e})$ and compatible with arbitrary base change. Define $\mathcal{G} := Gr(k, \mathcal{E})$. $\mathcal{G}$ is rational as a locally free fibre bundle with a Grassmannian as fibre over the rational variety $\mathcal{H}^0_H$. Let $\gamma : \mathcal{G} \to \mathcal{H}^0_H$ be the natural projection and consider the morphism

$$q := \gamma \times id_{X_e} : \mathcal{G} \times X_e \longrightarrow \mathcal{H}^0_H \times X_e.$$

By Corollary 4.5 of [16] over $\mathcal{G} \times X_e$ we have a tautological extension

$$0 \longrightarrow q^*(\vartheta_1) \longrightarrow \mathcal{V} \longrightarrow q^*(\vartheta_2) \otimes \mathcal{O}_{\mathcal{G}}(-1) \longrightarrow 0$$

such that for each $t \in \mathcal{G}$ the restriction $\mathcal{V}_t$ of $\mathcal{V}$ to $\{t\} \times X_e$ is isomorphic to the extension corresponding to t. Moreover, there is a natural bijective morphisms $\mathcal{G} \to \mathcal{F}$.

Note that any $E \in \mathcal{F}$ is a rank r vector bundle with Chern classes

$$(c_1(E), c_2(E)) = (C_0 - xF, -(r-k-1)f) = (C_0 - xF, c_2)$$

and by construction $h^0(E) \geq k$.

Claim. *E is H-stable.*

Proof of the Claim. We will proceed by induction on k. First of all assume $k = 1$. In that case E is given by a non-trivial extension

$$0 \to \mathcal{O}_{X_e} \to E \xrightarrow{\sigma} G \to 0$$

where G is a rank $r-1$ H-stable vector bundle. Let F be a subbundle of E. Denote by F_2 the image of F by the map σ and denote by F_1 the kernel of the induced map $\sigma : F \to F_2$. By construction F_1 is a subbundle of $\mathcal{O}_{X_e}$, F_2 is a subbundle of G and we have the exact sequence

$$0 \to F_1 \to F \xrightarrow{\sigma} F_2 \to 0.$$

Now we will distinguish two cases.
Case 1: Assume that $F_1 = 0$.

In that case, $F \cong F_2$ and it is a subbundle of G. Since, we have

$$c_1(E)H = (C_0 - xF)(C_0 + (x+e+1)F) = 1,$$

if F destabilizes E, we deduce

$$\frac{c_1(F)H}{rank(F)} \geq \frac{c_1(E)H}{rank(E)} = \frac{1}{r}.$$

On the other hand, F is a subbundle of the H-stable vector bundle G. Thus,

$$\frac{c_1(F)H}{rank(F)} < \frac{c_1(G)H}{rank(G)} = \frac{1}{r-1}.$$

Therefore,

$$\frac{1}{r} \leq \frac{c_1(F)H}{rank(F)} < \frac{1}{r-1}. \tag{3.6}$$

In particular, we have $c_1(F)H \geq 1$ and hence

$$\frac{1}{rank(F)} < \frac{c_1(F)H}{rank(F)} < \frac{1}{r-1}$$

which implies that $r-1 < rank(F)$ contradicting that $rank(F) \leq r-1$.
Case 2: Assume that $F_1 \neq 0$.

Since F_1 is a non-zero subbundle of the semistable bundle $\mathcal{O}_{X_e}$, F_1 is a rank one torsion free sheaf and $c_1(F_1)H \leq 0$. On the other hand, since F_2 is a subbundle of G which is H-stable,

$$\frac{c_1(F_2)H}{rank(F_2)} < \frac{c_1(G)}{r-1} = \frac{1}{r-1}$$

i.e

$$c_1(F_2)H < \frac{rank(F_2)}{r-1} \leq 1$$

which implies that $c_1(F_2)H \leq 0$. Altogether we get

$$\frac{c_1(F)H}{rank(F)} = \frac{(c_1(F_1) + c_1(F_2))H}{rank(F)} \leq 0 < \frac{1}{r} = \frac{c_1(E)H}{rank(E)}.$$

Therefore, E is H-stable.

Now, we fix $k > 1$ and we consider the following commutative diagram of vector bundles

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{X_e} & \longrightarrow & \mathcal{O}_{X_e}^{k} & \longrightarrow & \mathcal{O}_{X_e}^{k-1} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{X_e} & \longrightarrow & E & \longrightarrow & \overline{E} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G & = & G & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

By the induction hypothesis, $\overline{E}$ is H-stable and thus by the first case $k = 1$, E is H-stable which concludes the proof of the Claim. $\square$

Arguing as in the proof of Proposition 3.3, we get that $\mathcal{F}$ embeds into the Brill–Noether loci $W_H^k(r; C_0 - xF, c_2)$. Let us compute the dimension of $\mathcal{F}$. By definition

$$\dim \mathcal{F} = \dim \mathcal{H}_H^0(r-k; C_0 - xF, c_2) + \dim Gr(k, W)$$

being $h = ext^1(G, \mathcal{O}_{X_e})$ and $Gr(k, W)$ the Grassmannian variety of k-dimensional linear subspaces of $W = \mathrm{Ext}^1(G, \mathcal{O}_{X_e})$. Notice that by Serre duality $h = ext^1(\mathcal{O}_{X_e}, G(K_{X_e})) = h^1G(K_{X_e})$. By Proposition 3.3, G is given by an exact sequence of the following type

$$0 \to \mathcal{O}_{X_e}(C_0 - (x - (r-k-1)f)F) \to G \to \mathcal{O}_{X_e}(-fF)^{r-k-1} \to 0. \tag{3.7}$$

Twisting it by $\mathcal{O}_{X_e}(K_{X_e})$ and taking cohomology we get the long exact cohomological sequence

$$\begin{aligned}
&\to H^1\mathcal{O}_{X_e}(-C_0 - (x - (r-k-1)f + e + 2)F) \to H^1G(K_{X_e}) \to H^1\mathcal{O}_{X_e}(K_{X_e} - fF)^{r-k-1} \\
&\to H^2\mathcal{O}_{X_e}(-C_0 - (x - (r-k-1)f + e + 2)F).
\end{aligned}$$

By Remark 2.1 and Serre's duality we obtain

$$H^1\mathcal{O}_{X_e}(-C_0 - (x - (r-k-1)f + e + 2)F) = H^2\mathcal{O}_{X_e}(-C_0 - (x - (r-k-1)f + e + 2)F) = 0.$$

Therefore, by the above exact sequence and Lemma 2.2 we get

$$h^1G(K_{X_e}) = (r-k-1)h^1(\mathcal{O}_{X_e}(K_{X_e} - fF)) = (r-k-1)h^1\mathcal{O}_{X_e}(fF) = -(r-k-1)(f+1).$$

Hence, since by Proposition 3.3 we have that

$$\dim \mathcal{H}_H^0(r-k; C_0 - xF, c_2) = -2(r-k)(r-k-1)f + (r-k-1)(e+2x) - (r-k)^2 + 1,$$

we get

$$\begin{aligned}
\dim \mathcal{F} &= -2(r-k)(r-k-1)f + (r-k-1)(e+2x) - (r-k)^2 + 1 - k(r-k-1)(f+1) - k^2 \\
&= -2r(r-k-1)f - (r-k-1)(-e-2x) + k(r-k-1)f - r^2 + 1 - k^2 + k(r+1).
\end{aligned}$$

On the other hand, by definition

$$\rho_H^k(r; C_0 - xF, c_2) = \dim M_H(r; C_0 - xF, c_2) - k(k - \chi(r; C_0 - xF, c_2))$$

and by the Riemann–Roch theorem

$$\chi(r; C_0 - xF, c_2) = r - \frac{(C_0 - xF)K_{X_e}}{2} + \frac{(C_0 - xF)^2}{2} - c_2 = r - (e+2x) + 1 - c_2.$$

This together with the fact that by Proposition 2.4,

$$\dim M_H(r; C_0 - xF, c_2) = 2rc_2 - (r-1)(C_0 - xF)^2 - r^2 + 1,$$

implies that

$$\begin{aligned}
\rho_H^k(r; C_0 - xF, c_2) &= -2r(r-k-1)f - (r-1)(-e-2x) - r^2 + 1 - k^2 + k(r - (e+2x) + 1 + (r-k-1)f) \\
&= -2r(r-k-1)f - (r-k-1)(-e-2x) + k(r-k-1)f - r^2 + 1 - k^2 + k(r+1).
\end{aligned}$$

Therefore, the closure of $\mathcal{F}$ is an irreducible component of the Brill–Noether loci $W_H^k(r; C_0 - xF, c_2)$ of the moduli space $M_H(r; C_0 - xF, c_2)$ which has the expected dimension, namely, $\rho_H^k(r; C_0 - xF, c_2)$. Indeed, by [12–14,19], there exists a coarse moduli space $\mathcal{B}$ classifying pairs (E, V) where E is a rank r, H-stable vector bundle on X_e with Chern classes $(C_0 - xF, c_2)$ and $V \subset H^0(E)$ is a vector space of dimension k, which satisfy

(1) $ev : V \otimes \mathcal{O}_{X_e} \to E$ is an injection
(2) $\mathrm{Coker}(ev)$ is a rank $r - k$, H-stable vector bundle on X_e.

So, we have a correspondence

$$\begin{tikzcd} & \mathcal{B} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ M_H(r; C_0 - xF, c_2) & & M_H(r-k; C_0 - xF, c_2). \end{tikzcd}$$

A closed point $[(E, V)]$ in $\mathcal{B}$ gives us an exact sequence

$$0 \to \mathcal{O}_{X_e}^k \to E \to G \to 0$$

and

$$\begin{aligned} \dim \mathcal{B} &= \dim \pi_1^{-1}(E) + \dim(M_H(r; C_0 - xF, c_2)) \\ &= \dim \pi_2^{-1}(G) + \dim(M_H(r-k; C_0 - xF, c_2)). \end{aligned} \tag{3.8}$$

Since $\dim(M_H(r-k; C_0 - xF, c_2)) \geq \dim(W_H^k(r; C_0 - xF, c_2))$, if $\dim(W_H^k(r; C_0 - xF, c_2)) > \dim \mathcal{F}$, using (3.8) together with the above dimension computations, we would get $\dim(M_H(r-k; C_0 - xF, c_2)) > -2(r-k)(r-k-1)f + (r-k-1)(e+2x) - (r-k)^2 + 1 = \dim(M_H(r-k; C_0 - xF, c_2))$ which is a contradiction. $\square$

Remark 3.5. It is a long standing problem in vector bundle theory to give effective bounds for how large has to be c_2 in order to assure that the moduli space of stable vector bundles on surfaces is non-empty. It will be nice to address in the case of Hirzebruch surfaces whether a simple bound can be found.

4. Emptiness of the Brill–Noether loci of stable vector bundles on Hirzebruch surfaces

The goal of this last section is to prove the emptiness of a huge number of Brill–Noether loci (see Corollaries 4.2 and 4.3); these results were expected because the corresponding Brill–Noether numbers are negative. Our goal will be achieved as an easy consequence of the following simple but fundamental result which can be considered as a generalization of Clifford's Theorem for line bundles on curves.

Theorem 4.1. *Let X be a smooth algebraic surface, H an ample divisor on X such that $K_X H \leq 0$ and E a semistable rank $r \geq 2$ vector bundle on X. Set $a := \lceil \frac{(r^2-1)H^2}{2} \rceil$ or $a = 2H^2$ if $r = 2$. If $0 \leq c_1(E)H < arH^2 + rK_X H$, then*

$$h^0(E) \leq r + \frac{ac_1(E)H}{2}.$$

Proof. Let $C \in |aH|$ be a general smooth connected curve. Since

$$\frac{\binom{a+2}{2} - a - 1}{a} > deg(X) \cdot \max\left\{\frac{r^2-1}{4}, 1\right\},$$

by Flenser's restriction theorem [8], $E|_C$ is a rank r semistable vector bundle on C of degree equal to $ac_1(E)H$. On the other hand, by the adjunction formula

$$2g(C) - 2 = C(C + K_X) = aH(aH + K_X) = a^2H^2 + aHK_X.$$

Hence,

$$0 \leq \mu(E|_C) = \frac{ac_1(E)H}{r} \leq a^2H^2 + aHK_X = 2g(C) - 2$$

and therefore, applying Clifford's Theorem for semistable vector bundles on curves (see [2]; Theorem 2.1), we have

$$h^0(E|_C) \leq r + \frac{ac_1(E)H}{2}.$$

To finish the proof, we only need to check that

$$h^0(E) \leq h^0(E|_C).$$

To this end, we tensor by E the short exact sequence

$$0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$

and taking cohomology we get

$$0 \to H^0(E(-C)) \to H^0(E) \to H^0(E|_C) \to \cdots. \tag{4.1}$$

If $H^0(E(-C)) \neq 0$, then $\mathcal{O}_X(C) \hookrightarrow E$ and since E is semistable with respect to H we have

$$CH = aH^2 \leq \frac{c_1(E)H}{r} < aH^2 + K_X H$$

which contradicts the fact that $K_X H \leq 0$. Therefore, $H^0(E(-C)) = 0$ and from the exact sequence (4.1) we deduce $h^0(E) \leq h^0(E|_C)$ and we finish the proof. $\square$

As an application of the generalization of Clifford's theorem we obtain our first general result concerning the emptiness of the Brill–Noether loci. Indeed, as a Corollary of Theorem 4.1 we get

Corollary 4.2. *Let X be a smooth algebraic surface, H an ample divisor on X such that $K_X H \leq 0$. Let $r \geq 2$, $c_2 \gg 0$ be integers and set $a := \lceil \frac{(r^2-1)H^2}{2} \rceil$ or $a = 2H^2$ if $r = 2$. Assume that $0 \leq c_1(E)H < arH^2 + rK_X H$, then*

$$W_H^k(r; c_1, c_2) = \emptyset$$

for any $k > r + \frac{ac_1(E)H}{2}$.

We will now apply the above result to the particular case of moduli spaces of vector bundles on Hirzebruch surfaces studied in Section 3.

Corollary 4.3. *Let X_e be a Hirzebruch surface, x, r, c_2 integers with $x > 0$, $r \geq 2$ and $c_2 \gg 0$. Let $H := C_0 + (x + e + 1)F$ be an ample divisor on X_e and set $a := \lceil \frac{(r^2-1)H^2}{2} \rceil$ or $a = 2H^2$ if $r = 2$. Then, for any $k > r + \frac{a}{2}$*

$$W_H^k(r; C_0 - xF, c_2) = \emptyset.$$

Proof. First of all we observe that

$$c_1(E)H = (C_0 - xF)(C_0 + (x + e + 1)F) = 1$$

and

$$0 \leq c_1(E)H < arH^2 + rK_{X_e}H.$$

Thus, the result follows by Corollary 4.2. □

Remark 4.4. Notice that in all the above cases where we prove that the Brill–Noether loci $W_H^k(r; c_1, c_2)$ is empty, the expected dimension $\rho_H(r; c_1, c_2)$ is negative.

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